The Binomial Theorem

In elementary algebra, the binomial theorem describes the algebraic expansion of power of a binomial. According to the theorem, it is possible to expand the power of $(x+y)^n$ into a sum involving terms of the form ax^by^c, where the coefficient of each term is a positive integer (whole number), and the sum of the exponents of x and y in each term is n. For example:

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

The coefficients appearing in the binomial expansions are known as **binomial coefficients**. They are the same as the entries of Pascal's Triangle, and can be determined by a simple formula involving factorials.

Statement of the theorem

According to the theorem, it is possible to expand any power of $x+y$ into a sum of the form

$$(a+b)^n = \binom{n}{0}a^nb^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + ... + \binom{n}{n-1}a^1b^{n-1} + \binom{n}{n}a^0b^n$$

where $\binom{n}{r}$ denotes the correspongind binomial coefficient. Using the summation notation $(\sum)$, the formula above can be written as

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{r} a^{n-r} b^r$$

This formula is sometimes referred to as the **binomial formula** or the **binomial identity**.

Factorials

Before I begin with examples, two mathematical notation needs to be introduced first. The notation of 5! means $5 \times 4 \times 3 \times 2 \times 1$, also know as a **factorial**. Note that $1! = 1$ and that by convention $0! = 1$.

Binomial Coefficient

The second notation is the Binomial Coefficient

$$\binom{n}{r}$$

which is defined by

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Thus

$$\binom{8}{2} = \frac{8!}{2!6!} = \frac{8\times 7}{2} = 28$$

and

$$\binom{6}{3} = \frac{6!}{3!3!} = \frac{6\times 5\times 4}{6} = 20$$

Now that we have defined all the notations needed to use the Binomial formula, we can now move on to the examples.

Examples

Example 1 Use the binomial formula to expand $(2+x)^3$

Solution

Using the binomial formula

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{r} a^{n-r} b^r$$

$a = 2$, $b = x$ and $n = 3$. Then

$$\begin{aligned}(2+x)^3 &= \binom{3}{0}2^3 + \binom{3}{1}2^2x + \binom{3}{2}2x^2 + \binom{3}{3}x^3 \\ &= 8 + 12x + 6x^2 + x^3\end{aligned}$$

Example 2 Use the Binomial formla to expand $(\theta + \frac{1}{\theta})^4$

Solution

$$\begin{aligned}(\theta + \frac{1}{\theta})^4 &= \binom{4}{0}\theta^4 + \binom{4}{1}\theta^3(\frac{1}{\theta}) + \binom{4}{2}\theta^2(\frac{1}{\theta^2}) + \binom{4}{3}\theta(\frac{1}{\theta^3} + \binom{4}{4}(\frac{1}{\theta^4}) \\ &= \theta^4 + 4\theta^3\frac{1}{\theta} + 6\theta^2\frac{1}{\theta^2} + 4\theta\frac{1}{\theta^3} + \frac{1}{\theta^4} \\ &= \theta^4 + 4\theta^2 + 6 + 4\theta^{-2} + \theta^{-4}\end{aligned}$$