Victoria University of Wellington
School of Mathematics, Statistics & Operations Research
Math 434

Set Theory 2009

Assignment One

Due: 16 October 2009

(1). Let x, y be two sets and define

$$\langle x, y\rangle = \{\{x\}, \{x, y\}\}.$$

(a) Show that $\langle x, y\rangle = \langle u, v\rangle$ iff $x = u$ and $y = v$.

(b) If A is any set and $x, y \in A$ show that $\{x\}$ and $\{x, y\}$ are in $\wp(A)$; and $\langle x, y\rangle \in \wp(\wp(A))$.

(c) If A is a set, show that $B = \{\langle x, y\rangle \mid x, y \in A\}$ is also a set.

(d) If C is a set and $\{x, y\} \in C$ show that $x, y \in \bigcup C$.

(e) If D is a set and $\langle x, y\rangle \in D$ show that $x, y \in \bigcup\bigcup D$.

(2). Let $\mathbb{N} = \bigcap\{X \mid X \text{ is inductive}\}$. Let $0 = \emptyset$ and $n + 1 = n \cup \{n\}$. Define $<$ on $\mathbb{N}$ by

$$n < m \iff n \in m.$$

A set S is T-finite iff every nonempty $X \subseteq \wp(S)$ has a $\subset$-maximal element, i.e. $u \in X$ such that there is no $v \in X$ with $u \subset v$.
A set T is *transitive* iff $x \in T$ implies $x \subset T$.
Show :

(a) If X is inductive, then the set $\{x \in X \mid x \subseteq X\}$ is inductive. Hence $\mathbb{N}$ is transitive, and for each $n \in \mathbb{N}$

$$n = \{m \in \mathbb{N} \mid m < n\}.$$

(b) If X is inductive, then the set $\{x \in X \mid x \text{ is transitive}\}$ is inductive. Hence every $n \in \mathbb{N}$ is transitive.

(c) If X is inductive, then the set $\{x \in X \mid x \text{ is transitive and } x \notin x\}$ is inductive. Hence $n \notin n$ and $n \neq n + 1$ for each $n \in \mathbb{N}$.
[DO NOT use regularity to prove this.]

(d) $\mathbb{N}$ is T-infinite; the set $\mathbb{N} \subseteq \wp(\mathbb{N})$ has no $\subset$-maximal element.

(e) If X is inductive, then the set
$\{x \in X \mid x \text{ is transitive and every nonempty } z \subseteq x \text{ has an } \in\text{-minimal element}\}$ is inductive.
[t is $\in$-minimal in z if there is no $s \in z$ such that $s \in t$.]

(f) Every nonempty $X \subseteq \mathbb{N}$ has an $\in$-minimal element.
[Pick $n \in X$ and look at $X \cap n$].

(g) If X is inductive, then the set $\{x \in X \mid x = \emptyset \text{ or } x = y \cup \{y\} \text{ for some } y\}$ is inductive. Hence each $n \neq 0$ is $m + 1$ for some m.

(h) *Induction.* Let A ne a subset of $\mathbb{N}$ such that $0 \in A$ and $n \in A \to n + 1 \in A$. Then $A = \mathbb{N}$.

(i) Each $n \in \mathbb{N}$ is T-finite.

A set X is said to have n elements iff there is a bijection from n to X. A set is *finite* if it has n elements for some $n \in \mathbb{N}$.

(j) Every finite set is T-finite.

(k) Every infinite set is T-infinite.
[If S is infinite, consider $X = \{u \subset S \mid u \text{ is finite}\}$.]

(**3**). Let A be a set of ordinals.

(a) Show that $\bigcup A$ is an ordinal.

(b) Let β be any ordinal such that $\alpha \leq \beta$ for all $\alpha \in A$. Show that $\bigcup A \leq \beta$.

(c) Show that for any limit ordinal $\beta \quad \bigcup \beta = \beta$.

(d) Show that for any ordinal $\alpha \quad \bigcup(\alpha + 1) = \alpha$.

(**4**). Let OR_α denote the α^{th} class obtained via Cantor-Bendixon derivation.

(a) Show that $\mathrm{OR}_{a+1} \subseteq \mathrm{OR}_\alpha$ for all α.

(b) Prove, in detail, that $\omega^\alpha \in \mathrm{OR}_\alpha$ for all α.

(c) Show that $\beta \in \mathrm{OR}$ and $\gamma \in \mathrm{OR}_\alpha$ implies $\beta + \gamma \in \mathrm{OR}_\alpha$ for all α, β, γ.

(d) Let α, β and γ be any three ordinals. Show that $\beta + \gamma \in \mathrm{OR}_\alpha$ implies $\gamma \in \mathrm{OR}_\alpha$.

(e) Show that $\gamma \in \mathrm{OR}_\alpha \setminus \mathrm{OR}_{\alpha+1}$ iff there is an ordinal β such that $\gamma = \beta + \omega^\alpha$.
[Your proof should be based on the preceding results, not on the Cantor Decomposition theorem.]

(**5**). Let X be a set of ordinals. Define a function f on X by

$$f(x) = \{f(y) \mid y \in X \text{ and } y \in x\}.$$

(a) Show that $f(\min X) = \emptyset$;

(b) Show that $f(x)$ is an ordinal for all $x \in X$;

(c) Show that the range of f is an ordinal;

(d) Show that f is one-one.