Victoria University of Wellington
School of Mathematics, Statistics & Operations Research
Math 434

Set Theory 2009

Assignment Two

Due: 16 October 2009

(1). Let $\mathcal{P} = \langle P, \leq \rangle$ be any linearly ordered set. Let $\preceq$ be a well-ordering of P of length $|P|$. Define

$$B = \{q \in P \mid \forall p \in P \; q \leq p \implies q \preceq p\}.$$

(a) Let $r \in P$ and define

$$s = \min_{\preceq} \{p \in P \mid r \leq p\}.$$

Show that $s \in B$ and $r \leq s$.

(b) Show that $\preceq$ and $\leq$ agree on B – ie if $a, b \in B$ then

$$a \leq b \iff a \preceq b.$$

(c) Show that $\langle B, \preceq \rangle$ is a well-order with ordertype at most $|P|$.

(2).

(a) Show that the set of finite subsets of ω is countable.

(b) Show that the set of permutations of ω has size $2^{\aleph_0}$.

(3). Show the following:

(a) $2^\omega = \omega^\omega$;

(b) $(2^\omega)^\omega = 2^\omega$;

(c) $\omega_1^\omega = 2^\omega$.

(4). Suppose that $\kappa^2 = \kappa$ for all infinite cardinals. Show that $\kappa + \lambda = \kappa \cdot \lambda$ for all infinite cardinals – without using the Axiom of Choice!

(5). For infinite cardinals $\lambda \leq \kappa$ show that

$$|\{X \subseteq \kappa \mid |X| = \lambda\}| = \kappa^\lambda.$$

(6). When λ is an infinite cardinal and κ is any cardinal recall that

$$\kappa^{<\lambda} = \left| \bigcup_{\alpha<\lambda} {}^{\alpha}\kappa \right| .$$

(a) Show that $\kappa^{<\lambda} = \sup \left\{ \kappa^{\theta} \mid \theta < \lambda \text{ and } \theta \text{ is a cardinal} \right\}$.

(b) For any $\tau_0 < \lambda$ show that $\kappa^{<\lambda} = \sum_{\tau_0<\tau<\lambda} \kappa^{\tau}$.

(c) Show that either

(i) there is some $\tau < \lambda$ such that $\kappa^{<\lambda} = \kappa^{\tau}$; OR

(ii) there is a sequence of cardinals $\langle \lambda_\xi \mid \xi < \mathrm{cf}(\lambda) \rangle$ cofinal in λ such that – $\alpha < \beta$ implies $\kappa^{\lambda_\alpha} < \kappa^{\lambda_\beta}$, and $\kappa^{<\lambda} = \sum_{\xi} \kappa^{\lambda_\xi}$.

(d) Show that c(i) implies $\mathrm{cf}(\kappa^{<\lambda}) \geq \lambda$.

(e) Show that c(ii) implies $\mathrm{cf}(\kappa^{<\lambda}) = \mathrm{cf}(\lambda)$.

(f) Show that case c(i) implies

$$(\kappa^{<\lambda})^{\rho} = \begin{cases} \kappa^{<\lambda} & \text{if } 0 < \rho < \lambda \\ \kappa^{\rho} & \text{if } \rho \geq \lambda. \end{cases}$$

(g) Show that case c(ii) implies

$$(\kappa^{<\lambda})^{\rho} = \begin{cases} \kappa^{<\lambda} & \text{if } 0 < \rho < \mathrm{cf}(\lambda) \\ \kappa^{\lambda} & \text{if } \mathrm{cf}(\lambda) \leq \rho < \lambda \\ \kappa^{\rho} & \text{if } \rho \geq \lambda. \end{cases}$$