Victoria University of Wellington
School of Mathematics, Statistics and Operations Research
Math 434

Set Theory 2009

Assignment Three

Due: 16 October 2009

Definition 0.1. Let $\mathbb{P} = \langle P, \leq \rangle$ be a partial order.
A *cover* for $\mathbb{P}$ is a set $C \subseteq P$ such that for all $p \in P$ there is a $c \in C$ with $p \leq c$.
The *cofinality* of $\mathbb{P}$ is

$$\mathrm{cf}(\mathbb{P}) = \min \{ |C| \mid C \text{ is a cover for } \mathbb{P} \} .$$

The *true cofinality* of $\mathbb{P}$ is

$$\mathrm{tcf}(\mathbb{P}) = \min \{ \alpha \mid \alpha \in \mathrm{OR} \text{ and } \exists f \colon \alpha \to P \text{ with } f \text{ increasing, and } \mathrm{rng}(f) \text{ is a cover for } \mathbb{P} \} .$$

(1). Show that every partial order has a cofinality.

(2). Show that $\langle \omega \times \omega_1, \leq \rangle$ (where $\langle n, \alpha \rangle \leq \langle m, \beta \rangle$ iff $n \leq m$ and $\alpha \leq \beta$) does not have a true cofinality.

(3). Show that $\mathrm{tcf}(\alpha)$ exists for all ordinals α (with the usual order).

(4). Show that if $\mathrm{tcf}(\mathbb{P})$ exists, then

(a) it is a regular cardinal;

(b) and equals $\mathrm{cf}(\mathbb{P})$.

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Definition 0.2. Let α be any ordinal. A subset $C \subseteq \alpha$ is

i. *closed* if for any $X \subseteq C$, then $\bigcup X \in C \cup \{\alpha\}$;

ii. *unbounded* if C is a cover for α;

iii. *club* if it is both closed and unbounded.

(5). Show that for any α there is a club $C \subseteq \alpha$ with ordertype equal to the cofinality of α.

(6). Show that if $\alpha > \omega$ is a regular cardinal, and $\mathscr{C}$ is a set of clubs in α of size strictly less than α, then $\bigcap \mathscr{C}$ is club in α.

(7). Suppose that α is a regular cardinal, and $\langle C_\beta \mid \beta \in \alpha \rangle$ is a family of clubs in α. Let

$$\Delta_\beta C_\beta = \{ x \mid \forall \gamma \in x \; x \in C_\gamma \} .$$

Show that $\Delta_\beta C_\beta$ is a club in α.