Victoria University of Wellington
School of Mathematics, Statistics & Operations Research
Math 434

Set Theory 2009

Assignment Four

Due: 16 October 2009

First we recall the definition of a ultrafilters, ultraproducts & ultrapowers and some of the basic properties of these constructs.

Definition 1: An *ultrafilter* on a set I is a set $U \subseteq \wp(I)$ such that

(a) $I \in U$, $\emptyset \notin U$;
(b) if $A \in U$ and $B \in U$ then $A \cap B \in U$;
(c) if $A \in U$ and $A \subseteq B$ then $B \in U$;
(d) if $A \subseteq I$ then either $A \in U$ or $I \setminus A \in U$.

Definition 2: An ultrafilter U on I is *principal* iff there is some $i \in I$ such that $U = \{A \mid A \subseteq I \text{ and } i \in A\}$.

Definition 3: An *ultrafilter* on a set I is *κ-complete* iff if $X \subseteq U$ and $|X| < \kappa$ then $\bigcap X \in U$.

Definition 4: If $\langle \mathfrak{A}_i \mid i \in I\rangle$ are a family of $\mathcal{L}$-models (for some language $\mathcal{L}$), and U is an ultrafilter on I then the ultrapower

$$\mathfrak{A} = \prod \mathfrak{A}_i/U$$

is the model with universe

$$\prod_{i\in I} A_i/\equiv$$

where $\equiv$ is the equivalence relation on $\prod_{i\in I} A_i$ given by

$$f \equiv g \text{ iff } \{I \mid f(i) = g(i)\} \in U.$$

Relations are defined by

$$R_a^{\mathfrak{A}}([f_1], \ldots, [f_k]) \text{ iff } \left\{i \,\middle|\, R_a^{\mathfrak{A}_i}(f_1(i), \ldots, f_k(i))\right\} \in U$$

and

$$f_b^{\mathfrak{A}}([f_1], \ldots, [f_k]) = [i \mapsto f_b^{\mathfrak{A}_i}(f_1(i), \ldots, f_k(i))].$$

Definition 5: If $\mathfrak{A}$ is an $\mathcal{L}$-model, and U an ultrafilter on I, then the *ultrapower* is the same as the ultraproduct of $\langle \mathfrak{A}_i = \mathfrak{A} \mid i \in I\rangle$, denoted by $\mathrm{Ult}_U(\mathfrak{A})$.

Definition 6: The *canonical embedding* from A into $\mathrm{Ult}_U(\mathfrak{A})$ is the function $j\colon A \to \mathrm{Ult}_U(\mathfrak{A})$ given by

$$j(a) = [i \mapsto a].$$

A few useful facts:

i. if φ is a formula and $f_1, \ldots, f_k$ are in $\prod_{i\in I} A_i$ then

$$\prod \mathfrak{A}_i/U \models \varphi[[f_1], \ldots, [f_k]] \text{ iff } \{i \mid \mathfrak{A}_i \models \varphi[f_1(i), \ldots, f_k(i)]\} \in U.$$

ii. if σ is any sentence then

$$\prod \mathfrak{A}_i/U \models \sigma \text{ iff } \{i \mid \mathfrak{A}_i \models \sigma\} \in U.$$

iii. if σ is any sentence, then

$$\mathfrak{A} \models \sigma \text{ iff } \mathrm{Ult}_U(\mathfrak{A}) \models \sigma.$$

iv. if φ is a formula and $a_1, \ldots, a_k$ are in A then

$$\mathfrak{A} \models \varphi[a_1, \ldots, a_k] \text{ iff } \mathrm{Ult}_U(\mathfrak{A}) \models \varphi[j(a_1), \ldots, j(a_k)].$$

If we are careful it is possible to define ultrapowers of V. We cannot just take equivalence classes as these are not sets. Instead we define the "equivalence class" of $f\colon I \to V$ as

$$[f] = \{g \mid g \equiv f \text{ and } \forall x \ \mathrm{rank}(x) < \mathrm{rank}(g) \to \neg(x \equiv f)\}.$$

With this definition and some care, all of the above results about $\mathrm{Ult}_U(V)$ still hold.

(1). Show that an ultrafilter U on I is κ-complete iff for any set $X \subseteq \wp(I)$, if $|X| < \kappa$ and $\bigcup X \in U$ then $X \cap U \neq \emptyset$.

(2). Let $\langle M, \in \rangle$ be a transitive model of some fragment of ZFC with $\omega \subseteq M$. Let U be a nonprincipal ultrafilter on ω. Define $f_k\colon \omega \to \omega$ by $f_k(n) = \max(n-k, 0)$. Show that the sequence

$$\langle [f_k] \mid k \in \omega \rangle$$

is strictly descending in $\mathrm{Ult}_U(M)$.

(3). Let $\langle M, \in \rangle$ be a transitive model of some fragment of ZFC with $\kappa \subseteq M$. Let U be a nonprincipal ultrafilter on κ which is **not** ω_1-complete.

(a) Show that there is a partition $\{X_n \mid n \in \omega\}$ of κ with $X_n \notin U$ for all n.

(b) Define $f_k\colon \kappa \to \omega$ by $f_k(x) = \max(n-k, 0)$ for $x \in X_n$.
Show that the sequence

$$\langle [f_k] \mid k \in \omega \rangle$$

is strictly descending in $\mathrm{Ult}_U(M)$.

(4). Let U be an ω_1-complete ultrafilter on I. Show that the "$\in$" relation in $\mathrm{Ult}_U(V)$ is well-founded.

(5). From the last exercise, if U is an ω_1-complete ultrafilter on I, then we can collapse $\mathrm{Ult}_U(V)$ to a transitive model M_U, in which case we get an elementary embedding $\hat{\jmath}$ as the composite:

$$V \xrightarrow{j} \mathrm{Ult}_U(V) \xrightarrow{\pi} M_U.$$

If $x \in M_U$ we say that $f\colon I \to V$ *represents* x iff $\hat{\jmath}(f) = x$.

1. Show that every ordinal α is represented by some $f\colon I \to \mathrm{OR}$.
2. Show that if $\hat{\jmath}(\alpha) = \alpha$ for all $\alpha \in \mathrm{OR}$ then $\hat{\jmath}$ is the identity function.
   [Try induction on rank.]
3. Show that $\alpha \leq \hat{\jmath}(\alpha)$ for all $\alpha \in \mathrm{OR}$.
4. Let κ be the least ordinal such that $\kappa < \hat{\jmath}(\kappa)$.
   i. Show that $\kappa > \omega$.
   ii. Define $D \subseteq \wp(\kappa)$ by
   $$A \in D \text{ iff } \kappa \in \hat{\jmath}(A).$$
   Show that D is a κ-complete ultrafilter on κ.