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MATH 437 Gröbner Bases and Their Applications 2009

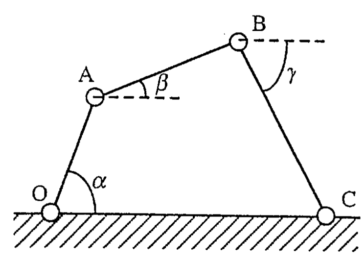
Assignment 1: Polynomials and Varieties

Due: Friday 24 July

1. Apply the method of Lagrange multipliers to find the maximum value of the quadratic form $f(x, y) = x^2 + y^2 + xy$ subject to the constraint $g(x, y) = x^2 + 2y^2 - 1 = 0$.

   Lagrange multipliers require you to solve the equations $\nabla f = \lambda \nabla g$ and $g = 0$ simultaneously. Here λ is a new variable introduced to solve the problem and ∇f is the gradient of f, i.e. $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$. If you can't do it by hand, try using Maple.

2. The simplest of planar parallel mechanisms is the *planar 4–bar*, consisting of four rigid components connected in a quadrilateral by revolute hinges. One bar is usually regarded as fixed. Convince yourself that, generally, such a mechanism will have one degree of freedom (which may include deciding what that actually means. . .).



   The configurations of a 4–bar can be described by means of the angles α, β, γ which are subject to equations which tell you that the quadrilateral is closed:

$$\begin{aligned} a\cos\alpha + b\cos\beta + c\cos\gamma - d &= 0 \\ a\sin\alpha + b\sin\beta + c\sin\gamma &= 0 \end{aligned}$$

   where a, b, c, d are the lengths of OA, AB, BC, CO respectively.

   Rewrite these equations as a system of polynomials in appropriate variables.

   See how far you can get in solving them in the case $a = b = c = d = 1$.

3. (a) Show that $g(x, y) = x^2y + xy^2 \in \mathbb{Z}_2[x, y]$ gives rise to the zero function on $\mathbb{Z}_2^2$.

(b) Find a non-zero polynomial in $\mathbb{Z}_2[x, y, z]$, involving all 3 variables, which vanishes identically on $\mathbb{Z}_2^3$.

(c) In the field $\mathbb{Z}_p$, $a^{p-1} = 1$ for all $a \in \mathbb{Z}_p$, $a \neq 0$. Deduce that $a^p = a$ for all $a \in \mathbb{Z}_p$ and hence find a non-zero polynomial in $\mathbb{Z}_p[x]$ which vanishes on $\mathbb{Z}_p$ as a function.

4. Use Theorem 1 in Lecture 2 to show, given $f, g \in k[x_1, \ldots, x_n]$ with k infinite, that $f = g$ if and only if the associated functions $\hat{f} = \hat{g}$. (This is easy.)

5. (a) (Theorem 1 in Lecture 2.) Given affine varieties $V = \mathbf{V}(f_1, \ldots, f_s)$ and $W = \mathbf{V}(g_1, \ldots, g_t)$, prove that $V \cap W$ and $V \cup W$ are also varieties. [Hint: for the union, consider polynomials of the form $f_i g_j$.]

   (b) Suppose $V \subset k^n$ and $W \subset k^m$ are varieties. Prove that $V \times W \subset k^{n+m}$ is a variety.

   (c) Prove that every finite set of points in k^n is a variety.

6. By thinking about the proof in Q5(a), identify $V((x-y)(x^2+4y^2-1), (z-1)(x^2+4y^2-1))$ as the union of two varieties and hence sketch the variety in $\mathbb{R}^3$.

7. The polar equation $r = \sin(2\theta)$ defines a four–leaved rose. Show that this curve is a variety as follows:

   (a) show that the rose is *contained in* the variety $V((x^2+y^2)^3 - 4x^2y^2)$;

   (b) show that the variety is contained in the rose.

   (c) you might try plotting the curve in Maple, first as a polar curve, then implicitly using the defining equation.

8. (a) Consider the set $X = \{(x, x) : x \in \mathbb{R}, x \neq 1\} \subseteq \mathbb{R}^2$. Prove that X is not a variety (i.e. $X \neq \mathbf{V}(f_1, \ldots, f_s)$ for any $f_i \in \mathbb{R}[x, y]$, $i = 1, \ldots, s$).

   (b) Hence or otherwise prove that: (i) an infinite union of varieties need not be a variety; (ii) the difference of two varieties need not be a variety.

9. Show that the 2–dimensional sphere $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$ has the rational parametrisation:

$$x = \frac{2u}{u^2+v^2+1}$$
$$y = \frac{2v}{u^2+v^2+1}$$
$$z = \frac{u^2+v^2-1}{u^2+v^2+1}$$

by considering where the line joining the "north" pole $(0, 0, 1)$ to the point $(u, v, 0)$ in the xy–plane meets the sphere again. (Show the line is defined parametrically by $(x, y, z) = (tu, tv, t-1)$.) You should draw a picture first to illustrate the construction.