VICTORIA UNIVERSITY OF WELLINGTON
SCHOOL OF MATHEMATICS, STATISTICS AND OPERATIONS RESEARCH
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MATH 437 GRÖBNER BASES AND THEIR APPLICATIONS 2009

Assignment 2: Ideals and Varieties

Due: Friday 31 July

1. (a) Prove that if $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$, then $\langle f_1, \ldots, f_s \rangle$ is an ideal.
   (b) Let $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$. Prove that $I(\mathbf{V}(f_1, \ldots, f_s))$ is an ideal.
2. Prove that $\langle f_1, \ldots, f_s \rangle \subseteq \mathbf{I}(V(f_1, \ldots, f_s))$.
3. (a) Prove that if $\langle f_1, \ldots, f_s \rangle = \langle g_1, \ldots, g_t \rangle$ then $\mathbf{V}(f_1, \ldots, f_s) = \mathbf{V}(g_1, \ldots, g_t)$ (where the f_i s and g_j s are in $k[x_1, \ldots, x_n]$).
   (b) Hence prove that $\mathbf{V}(x + xy, y + xy, x^2, y^2) = \mathbf{V}(x, y)$ in k^2.
4. Let $f \in k[x_1, \ldots, x_n]$ and suppose that $f \notin \langle x_1, \ldots, x_n \rangle$. Prove that $\langle x_1, \ldots, x_n, f \rangle = k[x_1, \ldots, x_n]$.
5. Let $f_1 = y - x^2$ and $f_2 = z - x^3$ in $\mathbb{R}[x, y, z]$. In Lecture 3, we showed that $V(f_1, f_2) = \{(t, t^2, t^3) \,|\, t \in \mathbb{R}\}$ and $I(\mathbf{V}(f_1, f_2)) = \langle f_1, f_2 \rangle$.
   (a) Use the parametrisation of V to show that $y^2 - xz \in I(\mathbf{V}(f_1, f_2))$.
   (b) Show directly that $y^2 - xz \in \langle f_1, f_2 \rangle$.
6. Now consider the set $W \subset \mathbb{R}^3$ parametrised by (t^2, t^3, t^4), $t \in \mathbb{R}$.
   (a) First show that W is a variety. You must find polynomials that vanish *if and only if* $(x, y, z) = (t^2, t^3, t^4)$.
   (b) Is $I(W)$ the same as the ideal generated by the polynomials you found in (a)?
7. Prove that if every ideal in $k[x_1, \ldots, x_n]$ is finitely generated then $k[x_1, \ldots, x_n]$ is Noetherian.
8. Let V, W be varieties. Prove that $V \subset W$ if and only if $I(V) \supset I(W)$. Hence show that $V = W$ if and only if $I(V) = I(W)$. Furthermore prove that if

$$V_1 \supseteq V_2 \supseteq \cdots \supseteq V_i \supseteq$$

is a descending sequence of affine varieties, then there is some $N \in \mathbb{N}$ such that $V_n = V_N$ for all $n \geq N$.

9. Let $f_1, f_2, \ldots$ be an infinite sequence of polynomials in $k[x_1, \ldots, x_n]$. Prove that for some N, the variety $\mathbf{V}(f_1, f_2, \ldots) = \mathbf{V}(f_1, \ldots, f_N)$ (where $\mathbf{V}(f_1, f_2, \ldots) = \{(a_1, \ldots, a_n) \in k^n \mid \forall i \in \mathbb{N},\ f_i(a_1, \ldots, a_n) = 0\}$).

10. Prove the uniqueness of q and r in the Division Algorithm.

11. Prove that for $f_1, f_2 \in k[x]$, $\langle f_1, f_2 \rangle = \langle \text{GCD}(f_1, f_2) \rangle$.

12. Find $\text{GCD}(f_1, f_2, f_3)$ where

$$\begin{aligned} f_1 &= x^5 - 2x^4 - x^2 + 2x \\ f_2 &= x^7 + x^6 - 2x^4 - 2x^3 + x + 1 \\ f_3 &= x^6 - 2x^5 + x^4 - 2x^3 + x^2 - 2x. \end{aligned}$$

13. Prove the Ideal Membership Test: $f \in \langle f_1, \ldots, f_s \rangle \subseteq k[x]$ if and only if $f \xrightarrow{g}_+ 0$ where $g = \text{GCD}(f_1, \ldots, f_s)$.