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MATH 437 Gröbner Bases and Their Applications 2009

Assignment 3: The Division Algorithm and Term Orders

Due: Tuesday 18 August

1. Determine $\text{LT}(f)$ for $f(x,y) = 3x^4z - 2x^3y^4 + 7x^2y^2z^3 - 8xy^3z^3 \in \mathbb{Q}[x,y,z]$ with respect to LEX, DEGLEX and DEGREVLEX and given $x > y > z$.

2. Write down the DEGLEX and DEGREVLEX orderings for the degree 4 monomials in $k[x_1, x_2, x_3]$ given $x_1 > x_2 > x_3$.

3. Prove that DEGREVLEX is a term order.

4. Prove that the only term order on $k[x]$ is the usual one:

$$1 < x < x^2 < x^3 < \dots$$

5. Prove the lemma that in any term order, $x^\alpha | x^\beta$ implies $x^\alpha \leq x^\beta$.

6. A polynomial $f \in k[x_1, \dots, x_n]$ is called **homogeneous** if every term has the same total degree (*eg* $x^2y^2z + 4xy^4 - 3zyz^3 - z^5$, having degree 5). Given f homogeneous and working with DEGREVLEX (with $x_1 > x_2 > \cdots > x_n$), prove that $x_n | f$ if and only if $x_n | \text{LT}(f)$. More generally, prove that $f \in \langle x_i, \dots, x_n \rangle$ if and only if $\text{LT}(f) \in \langle x_i, \dots, x_n \rangle$.

7. A polynomial $f \in k[x_1, \dots, x_n]$ is called *symmetric* if it is invariant (remains the same) under any permutation of $x_1, \dots, x_n$. For example, if $n = 3$, $x_1 + x_2 + x_3$, $x_1x_2 + x_1x_3 + x_2x_3$ and $x_1x_2x_3$ are all symmetric. In general the *elementary symmetric polynomials* are:

$$\begin{aligned} \sigma_1 &= x_1 + x_2 + \cdots + x_n \\ \sigma_2 &= x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n \\ &\dots \\ \sigma_n &= x_1x_2 \dots x_n. \end{aligned}$$

Use the following steps to prove:
Fundamental Theorem of Symmetric Polynomials. Every symmetric $f \in k[x_1, \dots, x_n]$ can be written in the form $f = h(\sigma_1, \dots, \sigma_n)$ where $h \in k[x_1, \dots, x_n]$.

(a) Furnish $k[x_1, \dots, x_n]$ with LEX ordering and $x_1 > x_2 > \cdots > x_n$. Given f symmetric, let $\text{LT}(f) = cx^\alpha$ where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and $c \in k$. Prove that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$.

(b) If

$$g = \sigma_1^{\alpha_1 - \alpha_2} \sigma_2^{\alpha_2 - \alpha_3} \dots \sigma_{n-1}^{\alpha_{n-1} - \alpha_n} \sigma_n^{\alpha_n}$$

show that $\text{LP}(g) = x^\alpha$.

(c) Observe that $\text{LP}(f - cg) < \text{LP}(f)$ and that $f - cg$ is symmetric. Now use the well-ordering of the term order to prove the existence of the required polynomial h and hence prove the Theorem.

(d) Apply the algorithm above to write $f = x_1^4 + x_2^4$ ($n = 2$) as a polynomial in $\sigma_1 = x_1 + x_2$ and $\sigma_2 = x_1 x_2$.

8. Compute a remainder for the given f following reduction by the given set G with the given term order. (You may want to use the division layout in Cox et al, p64.)

   (a) $f = x^7y^2 + x^3y^2 - y + 1$; $G = \{xy^2 - x, x - y^3\}$ using DEGREVLEX $x > y > z$;

   (b) $f = xy^2z^2 + xy - yz$; $G = \{x - y^2, y - z^3, z^2 - 1\}$ using LEX $x > y$.

9. In Assignment 2, Q6, we considered the variety $V(z - x^2, y^2 - x^3)$ parametrised by $\{(t^2, t^3, t^4) \,|\, t \in \mathbb{R}\}$. Use the Division Algorithm with an appropriate term order, to find $u_1, u_2 \in \mathbb{R}[x, y, z]$ so that $g = y^4 - z^3 \in \mathbf{I}(V) \subseteq \mathbb{R}[x, y, z]$ satisfies

$$g = u_1.(z - x^2) + u_2.(y^2 - x^3).$$