VICTORIA UNIVERSITY OF WELLINGTON
SCHOOL OF MATHEMATICS, STATISTICS AND OPERATIONS RESEARCH
Te Kura Mātai Tatauranga, Rangahau Pūnaha

MATH 437 GRÖBNER BASES AND THEIR APPLICATIONS 2009

Assignment 4: Gröbner Bases and Buchberger's Algorithm

Due: Tuesday 8 September

It's time to start thinking about your project/class presentation. This counts up to 20% towards your final mark. The completion date is not till 16 October, but the mid-term break may be good opportunity to start thinking about your topic. Here are some possibilities:

- graph-colouring algorithms
- geometric theorem proving
- applications to robotics
- computer-aided geometric design
- optimisation and integer programming
- applications to coding theory and graph theory
- dynamical systems and bifurcations

There are others and you are welcome to suggest something yourself. Discuss with me what you want to do before you start and I can probably provide you with relevant readings or references. I will also give you guidance on content and presentation.

1. Prove
   (a) that (4)⇒(1) in the characterisation theorem, *ie* if $\text{LT}(G) = \text{LT}(I)$ then $G = \{g_1, \ldots, g_s\} \subseteq I$ is a Gröbner basis for I;
   (b) that if $G = \{g_1, \ldots, g_s\}$ is a Gröbner basis fro I then $I = \langle g_1, \ldots, g_s \rangle$;
   (c) that the remainder after reduction $f \xrightarrow{G}_+ r$ is unique.
2. Let $G = \{g_1, \ldots, g_s\} \subset k[x]$. Prove that G is a Gröbner basis for $\langle g_1, \ldots, g_s \rangle$ if and only if for some $i = 1, \ldots, s$, $g_i = c.\text{GCD}(g_1, \ldots, g_s)$ where $0 \neq c \in k$. (In other words, any Gröbner basis of an ideal in the PID $k[x]$ must contain a non-zero multiple of the unique monic generator for the ideal.)

3. Compute the S–polynomial $S(f, g)$ for $f = 3x^2yz - xy^3$ and $g = xy^2 + z^2 \in \mathbb{Q}[x, y, z]$ with $x > y > z$ and
   (a) LEX ordering
   (b) DEGLEX ordering
   (c) DEGREVLEX ordering.
4. In the Multivariate Division Algorithm, it may be that $\text{LT}(f)$ is divisible by both $\text{LP}(f_1)$ and $\text{LP}(f_2)$. Suppose that $f \xrightarrow{f_1} h_1$ and $f \xrightarrow{f_2} h_2$. Show that $h_1 - h_2$ is a polynomial multiple of $S(f_1, f_2)$.
5. Use Buchberger's Theorem to show that $f_1 = x - y^2w$, $f_2 = y - zw$, $f_3 = z - w^3$, $f_4 = w^3 - w \in \mathbb{Q}[x, y, z, w]$ with LEX and $x > y > z > w$ is a Gröbner basis for the ideal they generate.
6. Find a Gröbner basis for $\langle x^2y + z, xz + y\rangle \subseteq \mathbb{Q}[x, y, z]$ with respect to DEGLEX and $x > y > z$.
7. Consider the polynomials $f_1 = x^2 + y^2 + 1$, $f_2 = x^2y + 2xy + x$. I recommend using Maple to do the necessary reductions.
   (a) Find a Gröbner basis $\hat{G}$ for $I = \langle f_1, f_2\rangle \subseteq \mathbb{Q}[x, y]$ with respect to LEX ordering and $x > y$.
   (b) Find a Gröbner basis G_5 for $I = \langle f_1, f_2\rangle \subseteq \mathbb{Z}_5[x, y]$ with respect to LEX ordering and $x > y$. (It isn't the same as $\hat{G}$ reduced modulo 5. In Maple, you can use the command `mod 5`, just as in writing, to reduce polynomial coefficients modulo 5.)