VICTORIA UNIVERSITY OF WELLINGTON
SCHOOL OF MATHEMATICS, STATISTICS AND OPERATIONS RESEARCH
Te Kura Mātai Tatauranga, Rangahau Pūnaha

MATH 437 GRÖBNER BASES AND THEIR APPLICATIONS 2009

Assignment 5: Elimination and Closure

Due: Friday 18 September

1. Find bases for the elimination ideals I_1, I_2 where

$$I = \langle x^2 + y^2 + z^2 - 4, x^2 + 2y^2 - 5, xz - 1 \rangle.$$

2. (a) Suppose $1 \leq l \leq n$. A term order on $k[x_1, \dots, x_n]$ is said to be of *l–elimination type* provided any monomial involving any of $x_1, \dots, x_l$ is greater than any monomial in $k[x_{l+1}, \dots, x_n]$. Prove the Generalised Elimination Theorem: if $I \subseteq k[x_1, \dots, x_n]$ is an ideal and G a Gröbner basis for I with respect to an l–elimination type term order, then $G \cap k[x_{l+1}, \dots, x_n]$ is a Gröbner basis for the lth elimination ideal $I \cap k[x_{l+1}, \dots, x_n]$.

   (b) Prove that the following order $>_l$ is a term order of l–elimination type: $x^\alpha >_l x^\beta$ if

$$\alpha_1 + \cdots + \alpha_l > \beta_1 + \dots \beta_l$$

   or

$$\alpha_1 + \cdots + \alpha_l = \beta_1 + \dots \beta_l \qquad \text{and} \qquad \alpha >_{degrevlex} \beta.$$

3. Consider the equations

$$\begin{aligned} t^2 + x^2 + y^2 + z^2 &= 0 \\ t^2 + 2x^2 - xy - z^2 &= 0 \\ t + y^3 - z^3 &= 0 \end{aligned}$$

   and let I be the ideal in $k[t, x, y, z]$ generated by the left-hand sides. In what follows I would prefer you to use Maple, with the `Groebner` package.

   (a) Find a Gröbner basis for I with LEX ordering, $t > x > y > z$. What is a Gröbner basis for the first elimination ideal? In Maple, the term order is called `plex`.

   (b) Repeat with an elimination order that eliminates t. In Maple, the term order is called `lexdeg`.

(c) Which is the "better" term order for elimination? Explain what your criteria are.

4. Consider the equations:

$$\begin{aligned} x^5 + \frac{1}{x^5} &= y \\ x + \frac{1}{x} &= z. \end{aligned}$$

   (a) Rewrite them in a form that enables you to work in $\mathbb{C}[x, y, z]$.
   (b) Find a basis for $I_1 \subset \mathbb{C}[y, z]$ and show that $I_2 = \{0\}$.
   (c) Use the Extension Theorem to show that each partial solution $c \in V(I_2) = \mathbb{C}$ extends to a solution in $V(I) \subset \mathbb{C}^3$.
   (d) Which partial solutions $(y, z) \in V(I_1) \subset \mathbb{R}^2$ extend to real solutions in $V(I) \subset \mathbb{R}^3$?

5. (a) Prove that the ideal I', defined in the proof of the Weak Nullstellensatz, is an ideal.
   (b) Suppose A is an $n \times n$ matrix with entries in the (algebraically closed) field k. Define a function

$$\alpha : k[x_1, \ldots, x_n] \to k[x'_1, \ldots, x'_n]$$

   by $\alpha(f) = f'$ where $f'(x') = f(Ax')$.

      i. Show that α is linear.
      ii. Show that α is in fact a ring homomorphism: i.e. $\alpha(1) = 1$ and $\alpha(f.g) = \alpha(f).\alpha(g)$.
      iii. What conditions on A ensure that α is an isomorphism (i.e. 1–1 and onto)?
      iv. Given an ideal $I \subseteq k[x_1, \ldots, x_n]$, is $\alpha(I)$ an ideal? What about $\alpha^{-1}(I')$ for an ideal $I' \subseteq k[x'_1, \ldots, x'_n]$?
      v. Do any of the above apply if the entries of A are themselves polynomials in $k[x_1, \ldots, x_n]$?

6. (a) Prove that $\sqrt{I}$ is an ideal and is radical.
   (b) Prove that I is radical if and only if $I = \sqrt{I}$.
   (c) Deduce that $\sqrt{\sqrt{I}} = \sqrt{I}$.

7. Given an ideal $I = \langle f_1, \ldots, f_s \rangle$ in $k[x_1, \ldots, x_n]$, it follows from the proof of the Strong Nullstellensatz that $f \in \sqrt{I}$ if and only if $1 \in \langle f_1, \ldots, f_s, 1 - yf \rangle \subseteq k[x_1, \ldots, x_n, y]$. The ideal-membership problem can be resolved by first calulating a Gröbner basis for the ideal.

   (a) Show that $f = yz - x^3 \in \sqrt{I}$ where $I = \langle x^4y^2 + z^2 - 4xy^3z - 2y^5z, x^2 + 2xy^2 + y^4 \rangle$. (You can use Maple if you like.)
   (b) Find the smallest m such that $f^m \in I$. (Membership can be determined by reducing with respect to a Gröbner basis and seeing if the remainder is zero or not.)